2022 FALL REAL ANALYSIS (I) @ NCTU APPL. MATH.
HOMEWORK 3

- Please answer the following questions in details, which means you need to state all theorems and all reasons you have been using.
- Please mark your name, student ID, and question numbers clearly on your answer sheet. The deadline to hand in the exercise is on October 21, 2022.

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(1) Let $a_k \in \mathbb{R}$ be constants and $E_k \subset \mathbb{R}^n$ be subsets, for $k = 1, 2, \ldots, N$. A simple function $f(x) = \sum_{j=1}^{N} a_k \chi_{E_k}(x)$ is measurable if and only if E_k are measurable sets for all $k = 1, 2, \ldots, N$.

(2) (**The Borel-Cantelli lemma**) Suppose $\{E_k\}_{k=1}^{\infty}$ is a collection of countably many measurable subsets of $\mathbb{R}^n$, and

$$\sum_{k=1}^{\infty} m(E_k) < \infty.$$

Let

$$\begin{aligned} E &= \{x \in \mathbb{R}^n : x \in E_k, \text{ for infinitely many } k\} \\ &= \limsup_{k\to\infty}(E_k). \end{aligned}$$

Prove that
(a) E is measurable,
(b) $m(E) = 0$.

(3) If f is integrable in $(0,1)$, show that $x^k f(x)$ is also integrable in $(0,1)$, for all $k \in \mathbb{N}$. Moreover, $\int_0^1 x^k f(x)\,dx \to 0$ as $k \to \infty$.

(4) Let $f(x,y)$, $0 \le x, y \le 1$ satisfy the following conditions: For each x, $f(x,y)$ is an integrable function of y, and $\frac{\partial f(x,y)}{\partial x}$ is a bounded function of (x,y). Show that $\frac{\partial f(x,y)}{\partial x}$ is a measurable function of y for each x and

$$\frac{d}{dx}\int_0^1 f(x,y)\,dy = \int_0^1 \frac{\partial}{\partial x} f(x,y)\,dy.$$

(5) Let $E \subset \mathbb{R}^n$ be any measurable subset, and f be a nonnegative measurable function defined on E. Let $(f_m)(E) := \int_E f(x)\,dx$. Show that
(a) f_m is a (Lebesgue) measure.[1]

[1] For example, when $f = 1$, the integration stands for the usual Lebesgue measure.

(b) If T is a measurable and one-to-one map from $\mathbb{R}^n$ to $\mathbb{R}^m$ with a measurable inverse T^{-1}. Show that $(f_m) \circ (T^{-1}(K)) = (f \circ T^{-1})(K)$, where $K \subset \mathbb{R}^m$ is any measurable subset of $\mathbb{R}^m$.[2]

(6) Let $\{f_k\}$ be a sequence of measurable functions on E. Show that $\sum_{k=1}^{\infty} f_k$ converges absolutely a.e. in E if $\sum_{k=1}^{\infty} \int_E |f_k| < \infty$.

[2]Consider the change of variables in your calculus course.